Boundedness of Commutators of Singular and Potential Operators in Generalized Grand Morrey Spaces and some applications

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Abstract

In the setting of homogeneous spaces (X, d, μ), it is shown that the commutator of Calderón-Zygmund type operators as well as commutator of potential operator with BMO function are bounded in generalized Grand Morrey space. Interior estimates for solutions of elliptic equations are also given in the framework of generalized grand Morrey spaces.

Keywords: Generalized grand Morrey space; Commutator; Calderón-Zygmund operator; potential operator; elliptic PDEs.
2000 MSC: Primary 42B20, Secondary 42B25, 42B35

1. Introduction

In 1992 T. Iwaniec and C. Sbordone [22], in their studies related with the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. A generalized version of them, $L^{p),\theta}(\Omega)$ appeared in L. Greco, T. Iwaniec and C. Sbordone [21]. Harmonic analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during last years due to various applications, we mention e.g. [2, 10, 14, 15, 16, 17, 24].

Recently in [39] there was introduced a version of weighted grand Lebesgue spaces adjusted for sets $\Omega \subseteq \mathbb{R}^n$ of infinite measure, where the integrability of $|f(x)|^{p-\varepsilon}$ at infinity was controlled by means of a weight, and there grand grand Lebesgue spaces were also considered, together with the study of classical operators of harmonic analysis in such spaces. Another idea of introducing "bilateral" grand Lebesgue spaces on sets of infinite measure was suggested in [31], where the structure of such spaces was investigated, not operators; the spaces in [31] are two parametrical with respect to the exponent p, with the norm involving $\sup_{p_1 < p < p_2}$.

Morrey spaces $L^{p,\lambda}$ were introduced in 1938 by C. Morrey [33] in relation to regularity problems of solutions to partial differential equations, and provided a useful tool in the regularity theory of PDE's (for Morrey spaces we refer to the books [19, 30], see also [37] where an overview of various generalizations may be found).

Recently, in the spirit of grand Lebesgue spaces, A. Meskhi [34, 35] introduced *grand Morrey spaces* (in [34] it was already defined on quasi-metric measure spaces with doubling measure) and obtained results on the boundedness of the maximal operator, Calderón-Zygmund singular operators and Riesz potentials. The boundedness of commutators of singular and potential operators in grand Morrey spaces was already treated by X. Ye [43]. Note that the "*grandification* procedure" was applied only to the parameter p.

This paper is a continuation of the work began in [36] and [28], where in the former the introduction of generalized grand Morrey spaces (in that paper they where called grand grand Morrey spaces) and the study of maximal and Calderón-Zygmund operators was done in the framework of the Euclidean spaces whereas in the latter paper the study of the boundedness of potential operators was done in the framework of generalized grand Morrey spaces in homogeneous and even in the nonhomogeneous case.

Notation:
d_X denotes the diameter of the X set;
$A \sim B$ for positive A and B means that there exists $c > 0$ such that $c^{-1}A \leqslant B \leqslant cA$;
$B(x, r) = \{y \in X : d(x, y) < r\}$;
$A \lesssim B$ stands for $A \leqslant CB$;
by c and C we denote various absolute positive constants, which may have different values even in the same line;
$\hookrightarrow$ means continuous imbedding;
$\fint_B f \, d\mu$ denotes the integral average of f, i.e. $\fint_B f \, d\mu := \frac{1}{\mu B} \int_B f \, d\mu$;
p' stands for the conjugate exponent $1/p + 1/p' = 1$.

2. Preliminaries

2.1. Spaces of homogeneous type

Let $X := (X, d, \mu)$ be a topological space with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X, \mu)$ and d is a quasimetric, i.e. it is a non-negative real-valued function d on $X \times X$ which satisfies the conditions:

(i) $d(x, y) = 0$ if and only if $x = y$;

(ii) there exists a constant $C_t > 0$ such that $d(x, y) \leqslant C_t[d(x, z) + d(z, y)]$, for all $x, y, z \in X$, and

(iii) there exists a constant $C_s > 0$ such that $d(x, y) \leqslant C_s \cdot d(y, x)$, for all $x, y \in X$.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the d-balls $B(x, r)$. Everywhere in the sequel we assume that all the balls have a finite measure, that is, $\mu B(x, r) < \infty$ for all $x \in X$ and $r > 0$ and that for every neighborhood V of $x \in X$, there exists $r > 0$ such that $B(x, r) \subset V$.

We say that the measure μ is *lower α-Ahlfors regular*, if

$$\mu B(x, r) \geqslant cr^{\alpha} \tag{1}$$

and *upper β-Ahlfors regular* (or, it satisfies the *growth condition of degree β*), if

$$\mu B(x, r) \leq cr^{\beta}, \tag{2}$$

where $\alpha, \beta, c > 0$ does not depend on x and r. When $\alpha = \beta$, the measure μ is simply called *α-Ahlfors regular*.

The condition

$$\mu B(x, 2r) \leqslant C_d \cdot \mu B(x, r), \quad C_d > 1 \tag{3}$$

on the measure μ with C_d not depending on $x \in X$ and $0 < r < d_X$, is known as the *doubling condition*.

Iterating it, we obtain

$$\frac{\mu B(x, R)}{\mu B(y, r)} \leqslant C_d \left(\frac{R}{r}\right)^{\log_2 C_d}, \quad 0 < r \leqslant R \tag{4}$$

for all d-balls $B(x, R)$ and $B(y, r)$ with $B(y, r) \subset B(x, R)$.

The triplet (X, d, μ), with μ satisfying the doubling condition, is called a *space of homogeneous type*, abbreviated from now on simply as SHT. For some important examples of an SHT we refer e.g. to [7].

From (4) it follows that every homogeneous type space (X, d, μ) with finite measure is lower $(\log_2 C_d)$-Ahlfors regular.

Throughout the paper we will also assume the following condition

$$\mu(B(x, R) \backslash B(x, r)) > 0 \tag{5}$$

for all $x \in X$ and r, R with $0 < r < R < d_X$. The validity of the reverse doubling condition, following from the doubling condition under certain restrictions, is well known (cf., for example, [42, p. 269]). For example, when (5) is valid and (X, d, μ) is an SHT, then the measure μ also satisfies the reverse doubling condition

$$\frac{\mu B(x, r)}{\mu B(x, R)} \leqslant C \left(\frac{r}{R}\right)^{\gamma} \tag{6}$$

for appropriate positive constants C and γ. For other conditions dealing with the validity of the reverse doubling condition whenever the measure is doubling, see, e.g. [38].

2.2. Generalized Lebesgue spaces

For $1 < p < \infty$, $\theta > 0$ and $0 < \varepsilon < p - 1$ the *grand Lebesgue space* is the set of measurable functions for which

$$\|f\|_{L^{p),\theta}(X,\mu)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}(X,\mu)} < \infty, \tag{7}$$

where $\|f\|_{L^p(X,\mu)}^p := \int_X |f(y)|^p \, d\mu(y)$. In the case $\theta = 1$, we denote $L^{p),\theta}(X, \mu) := L^{p)}(X, \mu)$.

When $\mu X < \infty$, then for all $0 < \varepsilon < p - 1 < \theta_1 < \theta_2$ we have

$$L_w^p(X, \mu) \hookrightarrow L_w^{p),\theta_1}(X, \mu) \hookrightarrow L_w^{p),\theta_2}(X, \mu) \hookrightarrow L_w^{p-\varepsilon}(X, \mu),$$

where w is a Muckenhoupt weight.

For more properties of grand Lebesgue spaces, see [24].

2.3. Morrey spaces

For $1 \leqslant p < \infty$ and $0 \leqslant \lambda < 1$, the usual Morrey space $L^{p,\lambda}(X,\mu)$ is introduced as the set of all measurable functions such that

$$\|f\|_{L^{p,\lambda}(X,\mu)} := \sup_{\substack{x \in X \\ 0 < r < d_X}} \left(\frac{1}{\mu B(x,r)^\lambda} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \right)^{\frac{1}{p}} < \infty. \tag{8}$$

2.4. BMO space

The space of *bounded mean oscillation*, denoted by $\mathrm{BMO}(X,\mu)$, is the set of all real-valued locally integrable functions such that

$$\|f\|_{\mathrm{BMO}(X,\mu)} = \sup_{\substack{x \in X \\ 0 < r < d_X}} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y) < \infty, \tag{9}$$

where $f_{B(x,r)}$ is the integral average over the ball $B(x,r)$. $\mathrm{BMO}(X,\mu)$ is a Banach space with respect to the norm $\|\cdot\|_{\mathrm{BMO}}(X,\mu)$ when we regard the space $\mathrm{BMO}(X,\mu)$ as the class of equivalent functions modulo additive constants.

Remark 2.1. In this remark, we give equivalent norms for $\mathrm{BMO}(X,\mu)$-functions, namely

(i) we can define an equivalente norm in $\mathrm{BMO}(X,\mu)$ as

$$\|f\|_{\mathrm{BMO}(X,\mu)} \sim \sup_{\substack{x \in X \\ 0 < r < d_X}} \inf_{c \in \mathbb{R}} \frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - c| \, d\mu(y), \tag{10}$$

(ii) the John-Nirenberg inequality give us another equivalent norm for $\mathrm{BMO}(X,\mu)$-functions given by

$$\|f\|_{\mathrm{BMO}(X,\mu)} \sim \sup_{\substack{x \in X \\ 0 < r < d_X}} \left(\frac{1}{\mu B(x,r)} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p \, d\mu(y) \right)^{\frac{1}{p}} \tag{11}$$

valid for $1 < p < \infty$, where f_B stands for the integral average.

2.5. Maximal operators

We denote by Mf the *Hardy-Littlewood maximal operator*, given by

$$Mf(x) = \sup_{0 < r < d} \fint_{B(x,r)} |f(y)| \, d\mu(y), \tag{12}$$

for $x \in X$.

From [35] we have the following boundedness result for Morrey spaces.

Lemma 2.2. *Let $1 < p < \infty$ and $0 \leqslant \lambda < 1$. Then*

$$\|Mf\|_{L^{p,\lambda}(X,\mu)} \leqslant \left(Cb^{\lambda/p}(p')^{\frac{1}{p}} + 1 \right) \|f\|_{L^{p,\lambda}(X,\mu)}$$

holds, where the constant $b \geqslant 1$ arises in the doubling condition for μ and C is the constant independent of p.

4

By $M_s f$ we define the following *maximal operator*

$$M_s f(x) := (M|f|^s)^{\frac{1}{s}}$$

for $1 \leqslant s < \infty$. Using Lemma 2.2, it is easy to obtain that the following boundedness result.

Lemma 2.3. *Let* $1 < s < p < \infty$ *and* $0 \leqslant \lambda < 1$. *Then*

$$\|M_s f\|_{L^{p,\lambda}(X,\mu)} \leqslant \left(Cb^{\lambda s/p}((p/s)')^{\frac{s}{p}} + 1 \right) \|f\|_{L^{p,\lambda}(X,\mu)} .$$

holds, where the constant $b \geqslant 1$ *arises in the doubling condition for* μ *and* C *is the constant independent of* p.

When $\lambda = 0$, we have $L^p(X, \mu)$ boundedness of $M_s f$.

We will also need another maximal operator, namely the so-called *sharp maximal operator*,

Definition 2.4 (Sharp maximal function)**.** For all locally integrable function f and $x \in X$, we denote the sharp maximal function $f^\sharp(x)$ by

$$f^\sharp(x) = \sup_{0 < r < d_x} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}|\, d\mu(y).$$

It is immediate from the definition of the sharp maximal function that it is a.e. pointwise dominated by the maximal function, $f^\sharp(x) \leqslant 2Mf(x)$, but we also have some relation in the other direction, namely we have the so-called *Fefferman-Stein inequality* proved in the case of Lebesgue spaces in the Euclidean setting in [13]. Our version is taken from [32], namely

Lemma 2.5. *Let* $1 < p < \infty$ *and let* $0 \leqslant \lambda < 1$. *Then*

$$\|Mf\|_{L^{p,\lambda}(X,\mu)} \leqslant C(b^{\lambda/p} + 1) \left\| f^\sharp \right\|_{L^{p,\lambda}(X,\mu)} .$$

2.6. Calderón-Zygmund singular operators

We follow [35] in this section, in particular, making use of the following definition of the Calderón-Zygmund singular operators. Namely, the Calderón-Zygmund operator is defined as the integral operator

$$Tf(x) = \text{p.v.} \int_X K(x, y)f(y)\, d\mu(y)$$

with the kernel $K : X \times X \backslash \{(x, x) : x \in \Omega\} \to \mathbb{R}$ being a measurable function satisfying the conditions:

(i) $|K(x, y)| \leqslant \frac{C}{\mu B(x, d(x,y))}, \quad x, y \in X, \quad x \neq y;$

(ii) $|K(x_1, y) - K(x_2, y)| + |K(y, x_1) - K(y, x_2)| \leqslant Cw\left(\dfrac{d(x_2, x_1)}{d(x_2, y)} \right) \dfrac{1}{\mu B(x_2, d(x_2, y))}$

for all x_1, x_2 and y with $d(x_2, y) \geqslant Cd(x_1, x_2)$, where w is a positive non-decreasing function on $(0, \infty)$ which satisfies the Δ_2 condition $w(2t) \leqslant cw(t)$ $(t > 0)$ and the Dini condition $\int_0^1 w(t)/t\, dt < \infty$. We also assume that Tf exists almost everywhere on X in the principal value sense for all $f \in L^2(X)$ and that T is bounded in $L^2(X)$.

The boundedness of such Calderón-Zygmund operators in Morrey spaces is valid, as can be seen in the following Proposition, proved in [35].

5

Proposition 2.6. *Let $1 < p < \infty$ and $0 \leqslant \lambda < 1$. Then*

$$\|Tf\|_{L^{p,\lambda}(X,\mu)} \leqslant C_{p,\lambda}\|f\|_{L^{p,\lambda}(X,\mu)}$$

where

$$C_{p,\lambda} \leqslant c \begin{cases} \frac{p}{p-1} + \frac{p}{2-p} + \frac{p-\lambda+1}{1-\lambda} & \text{if } 1 < p < 2, \\ p + \frac{p}{p-2} + \frac{p-\lambda+1}{1-\lambda} & \text{if } p > 2, \end{cases} \tag{13}$$

with c not depending on p and λ.

2.7. Commutators

Let U be an operator and b a locally integrable function. We define the *commutator* $[b, U]f$ as

$$[b, U]f = bU(f) - U(bf).$$

Commutators are very useful when studying problems related with regularity of solutions of elliptic partial differential equations of second order, e.g., [4], [5]

3. Generalized grand Morrey spaces and the reduction lemma

In this section we will assume that the measure μ is upper γ-Ahlfors regular. All the stated results in this section were proved in [28].

We introduce the following functional

$$\Phi_{\varphi,A}^{p,\lambda}(f, s) := \sup_{0 < \varepsilon < s} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon,\lambda-A(\varepsilon)}(X,\mu)}, \tag{14}$$

where s is a positive number and A is a non–negative function defined on $(0, p - 1)$.

Definition 3.1 (Generalized grand Morrey spaces). Let $1 < p < \infty$, $0 \leqslant \lambda < 1$, φ be a positive bounded function with $\lim_{t \to 0+} \varphi(t) = 0$ and A be a non-decreasing real-valued non-negative function with $\lim_{x \to 0+} A(x) = 0$. By $L_{\varphi,A}^{p),\lambda}(X,\mu)$ we denote the space of measurable functions having the finite norm

$$\|f\|_{L_{\varphi,A}^{p),\lambda}(X)} := \Phi_{\varphi,A}^{p,\lambda}(f, s_{\max}), \quad s_{\max} = \min\{p - 1, a\}, \tag{15}$$

where $a = \sup\{x > 0 : A(x) \leqslant \lambda\}$.

Remark 3.2. For appropriate φ, in the case $A \equiv 0, \lambda > 0$ we recover the Grand Morrey spaces introduced in A. Meskhi [35], and when $\lambda = 0$, $A \equiv 0$ we have the grand Lebesgue spaces introduced in [21] (and in [22] in the case $\theta = 1$).

For fixed $p, \lambda, \varphi, A, f$ we have that $s \mapsto \Phi_{\varphi,A}^{p,\lambda}(f, s)$ is a non-decreasing function, but it is possible to estimate $\Phi_{\varphi,A}^{p,\lambda}(f, s)$ via $\Phi_{\varphi,A}^{p,\lambda}(f, \sigma)$ with $\sigma < s$ as follows.

Lemma 3.3. *For $0 < \sigma < s < s_{\max}$ we have that*

$$\Phi_{\varphi,A}^{p,\lambda}(f, s) \leqslant C\varphi(\sigma)^{-\frac{1}{p-\sigma}} \Phi_{\varphi,A}^{p,\lambda}(f, \sigma), \tag{16}$$

where C depends on γ, the parameters p, λ, φ, A and the diameter d_X, but does not depend on f, s and σ.

From Lemma 3.3 we immediately have

Lemma 3.4. *For $0 < \sigma < s_{\max}$, the norm defined in (15) has the following dominant*

$$\|f\|_{L^{p),\lambda}_{\varphi,A}(X)} \leqslant C \frac{\Phi^{p,\lambda}_{\varphi,A}(f,\sigma)}{\varphi(\sigma)^{\frac{1}{p-\sigma}}}, \tag{17}$$

where C depends on γ, the parameters p, λ, φ, A and the diameter d_X, but does not depend on f and σ.

Lemma 3.5 (Extended reduction lemma). *Let U and Λ be operators (not necessarily sublinears) satisfying the following relation in Morrey spaces*

$$\|Uf\|_{L^{q-\varepsilon,\lambda-A_2(\varepsilon)}(X)} \leqslant C_{p-\varepsilon,\lambda-A_1(\varepsilon),q-\varepsilon,\lambda-A_2(\varepsilon)}\|\Lambda f\|_{L^{p-\varepsilon,\lambda-A_1(\varepsilon)}(X)} \tag{18}$$

for all sufficiently small $\varepsilon \in (0,\sigma]$, where $0 < \sigma < s_{\max}$. If

$$\sup_{0<\varepsilon<\sigma} C_{p-\varepsilon,\lambda-A_1(\varepsilon),q-\varepsilon,\lambda-A_2(\varepsilon)} < \infty \tag{19}$$

and

$$\sup_{0<\varepsilon<\sigma} \frac{\psi(\varepsilon)^{\frac{1}{q-\varepsilon}}}{\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}} < \infty, \tag{20}$$

then the relation is also valid in the generalized grand Morrey space

$$\|Uf\|_{L^{q),\lambda}_{\psi,A_2}(X)} \leqslant C\|\Lambda f\|_{L^{p),\lambda}_{\varphi,A_1}(X)} \tag{21}$$

with

$$C = \frac{C_0}{\varphi(\sigma)^{\frac{1}{p-\sigma}}} \sup_{0<\varepsilon<\sigma} C_{p-\varepsilon,\lambda-A_1(\varepsilon),q-\varepsilon,\lambda-A_2(\varepsilon)},$$

where C_0 may depend on $\gamma, p, \lambda, \varphi, A$ and d_X, but does not depend on σ and f.

Proof. The proof follows the same lines as for the case where Λ is the identity operator, see [28], [29] for that proof. $\qquad\square$

Using the reduction lemma we obtain the boundedness of maximal and Calderón-Zygmund operators in generalized grand Morrey spaces, namely.

Theorem 3.6. *Let $1 < p < \infty$ and $0 \leqslant \lambda < 1$. Then the Hardy-Littlewood maximal operator is bounded from $L^{p),\lambda}_{\varphi,A}(X,\mu)$ to $L^{p),\lambda}_{\psi,A}(X,\mu)$ if there exists small σ such that $\sup_{0<\varepsilon<\sigma} \psi(\varepsilon)^{\frac{1}{q-\varepsilon}}/\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} < \infty$.*

Theorem 3.7. *Let $1 < p < \infty$, $\theta > 0$ and let $0 < \lambda < 1$. Then the Calderón-Zygmund operator T is bounded in the generalized grand Morrey spaces $L^{p),\lambda}_{\theta,A}(X,\mu)$.*

7

4. Boundedness of Commutators in Generalized Grand Morrey spaces

4.1. Commutator of Calderón-Zygmund operators

Before proving the main result in this subsection, we need some auxiliary results. The following lemma was proved in [43] but we give the proof for completeness of presentation.

Lemma 4.1. *Let T be a Caldéron-Zygmund operator, $0 < s < \infty$, if $b \in \mathrm{BMO}(X, \mu)$, then there exist a constant $C > 0$ such that for all functions f with compact support,*

$$([b, T]f)^{\sharp}(x) \leqslant C \|b\|_{\mathrm{BMO}(X,\mu)} \left(M_s(Tf)(x) + M_s(f)(x) \right). \tag{22}$$

Proof. For any ball $B = B(x, r) \subset X$, we write

$$
\begin{aligned}
[b, T]f(y) &= [b - b_{2B}, T]f(y) \\
&= (b - b_{2B})Tf(y) - T((b - b_{2B})f\chi_{2B})(y) - T\left((b - b_{2B})f\chi_{(2B)^{\complement}}\right)(y) \\
&= I_1(y) - I_2(y) - I_3(y).
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
\fint_B |[b, T]f(y) - I_3(z)| \, d\mu(y) &\leqslant \fint_B |b(y) - b_{2B}||Tf(y)| \, d\mu(y) + \fint_B |T((b - b_{2B})f\chi_{2B})(y)| \, d\mu(y) \\
&\quad + \fint_B \left| T\left((b - b_{2B})f\chi_{(2B)^{\complement}}\right)(y) - T\left((b - b_{2B})f\chi_{(2B)^{\complement}}\right)(z) \right| \, d\mu(y) \\
&:= \mathscr{I}_1(x) + \mathscr{I}_2(x) + \mathscr{I}_3(x, z).
\end{aligned}
$$

The estimation $\mathscr{I}_1(x) \leqslant C \|b\|_{\mathrm{BMO}(X,\mu)} M_s(Tf(x))$ is obtained via Hölder's inequality.

To estimate $\mathscr{I}_2$, there exists $s_0, s_1 > 1$, such that $1/s_0 = 1/s_1 + 1/s$, by Hölder's inequality, the L^s boundedness of T and the Remark 2.1 we have

$$
\begin{aligned}
\mathscr{I}_2(x) &\leqslant \left(\fint_B |T((b - b_{2B})f\chi_{2B})(y)|^{s_0} \, d\mu(y) \right)^{1/s_0} \\
&\lesssim \left(\fint_{2B} |b(y) - b_{2B}|^{s_0} |f(y)|^{s_0} \, d\mu(y) \right)^{1/s_0} \\
&\lesssim \left(\fint_B |b(y) - b_{2B}|^{s_1} \, d\mu(y) \right)^{1/s_1} \left(\fint_B |f(y)|^{s} \, d\mu(y) \right)^{1/s} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} M_s(f(x)).
\end{aligned}
$$

For any $z \in B$ and $y \in (2B)^{\complement}$, $2d(z, x) \leqslant d(y, x)$; it follows from (ii) of the definition of Calderón-

Zygmund operator that

$$
\begin{aligned}
\mathscr{I}_3(x) &\lesssim \fint_B \int_{(2B)^c} |K(z,y) - K(x,y)||b(y) - b_{2B}||f(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \\
&\lesssim \fint_B \int_{(2B)^\complement} w\left(\frac{d(z,x)}{d(y,x)}\right) \frac{1}{\mu B(x, d(x,y))} |b(y) - b_{2B}||f(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \\
&\lesssim \sum_{k=1}^{\infty} w\left(2^{-k}\right) \frac{1}{\mu B(x, 2^k r)} \int_{2^{k+1}B} |b(y) - b_{2B}||f(y)|\,\mathrm{d}\mu(y) \\
&\lesssim \sum_{k=1}^{\infty} w\left(2^{-k}\right) \left[\frac{1}{\mu B(x, 2^{k+1}r)} \int_{2^{k+1}B} |b(y) - b_{2B}|^{s'}\,\mathrm{d}\mu(y)\right]^{\frac{1}{s'}} \left[\frac{1}{\mu B(x, 2^{k+1}r)} \int_{2^{k+1}B} |f(y)|^s\,\mathrm{d}\mu(y)\right]^{\frac{1}{s}} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} M_s(f)(x) \sum_{k=1}^{\infty} w\left(2^{-k}\right),
\end{aligned}
$$

since $w(t)$ is a positive non-decreasing function on $(0, \infty)$ and satisfies the Dini condition,

$$
\sum_{k=1}^{\infty} w(2^{-k}) \leqslant c \int_0^1 \frac{w(t)}{t}\,\mathrm{d}t < \infty.
$$

Therefore, we have $\mathscr{I}_3(x) \lesssim \|b\|_{\mathrm{BMO}(X,\mu)} M_s(f)(x)$. Thus

$$
\begin{aligned}
([b,T]f)^\sharp(x) &= \sup_{0<r<d} \inf_{a \in R} \fint_{B(x,r)} |[b,T]f(y) - a|^s\,\mathrm{d}\mu(y) \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} (M_s(Tf)(x) + M_s(f)(x)). \qquad \square
\end{aligned}
$$

Lemma 4.2. *Let* $1 < p < \infty$, $0 < \lambda < 1$, *φ and A as in the definition of generalized grand Morrey spaces. If* $b \in \mathrm{BMO}(X,\mu)$*, then we have*

(i) $\|M([b,T]f)\|_{L_{\theta,A}^{p),\lambda}(X,\mu)} \lesssim \left\|([b,T]f)^\sharp\right\|_{L_{\theta,A}^{p),\lambda}(X,\mu)}$;

(ii) $\left\|([b,T]f)^\sharp\right\|_{L_{\theta,A}^{p),\lambda}(X,\mu)} \lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \left(\|M_s(Tf)\|_{L_{\theta,A}^{p),\lambda}(X,\mu)} + \|M_s f\|_{L_{\theta,A}^{p),\lambda}(X,\mu)}\right)$,

where M is the maximal operator and T is the Calderón-Zygmund operator.

Proof. The result of (i) follows at once taking into account Lemmas 2.5 and 3.5. For (ii), we simply use the pointwise estimate (22). $\square$

Theorem 4.3. *Let* $1 < p < \infty$, $\theta > 0$ *and* $0 < \lambda < 1$*. Suppose T is a Calderón-Zygmund operator and* $b \in \mathrm{BMO}(X,\mu)$*. Then the commutator $[b,T]$ is bounded in* $L_{\theta,A}^{p),\lambda}(X,\mu)$*.*

Proof. For any $1 < p < \infty$, $\theta > 0$ and $0 \leqslant \lambda < 1$, there exists an s such that $1 < s < p < \infty$ and

9

sufficiently small positive number σ. By above lemmas, we have

$$
\begin{aligned}
\|[b,T]f\|_{L^{p),\lambda}_{\theta,A}(X,\mu)} &\leqslant \|M([b,T]f)\|_{L^{p),\lambda}_{\theta,A}(X,\mu)} \\
&\leqslant C \sup_{0<\varepsilon\leqslant\sigma} \left(b^{(\lambda-A(\varepsilon))/(p-\varepsilon)}+1\right)\left\|([b,\ T]f)^{\#}\right\|_{L^{p),\lambda}_{\theta,A}(X,\mu)} \\
&\leqslant C \left(\|M_s(Tf)\|_{L^{p),\lambda}_{\theta,A}(X,\mu)} + \|M_s f\|_{L^{p),\lambda}_{\theta,A}(X,\mu)}\right) \\
&\leqslant C \sup_{0<\varepsilon\leqslant\sigma} \left(b^{\lambda s/(p-\varepsilon)}\left(\left(\frac{p-\varepsilon}{s}\right)'\right)^{\frac{s}{p-\varepsilon}}+1\right)\left(\|Tf\|_{L^{p),\lambda}_{\theta,A}(X,\mu)} + \|f\|_{L^{p),\lambda}_{\theta,A}(X,\mu)}\right) \\
&\leqslant C \left(\|Tf\|_{L^{p),\lambda}_{\theta,A}(X,\mu)} + \|f\|_{L^{p),\lambda}_{\theta,A}(X,\mu)}\right) \\
&\leqslant C \|f\|_{L^{p),\lambda}_{\theta,A}(X,\mu)}.
\end{aligned}
$$

$\square$

4.2. Commutators of potential operators

Let $0 < \alpha < 1$ and let

$$
I^\alpha f(x) = \int_X \frac{f(y)}{\mu B(x,d(x,y))^{1-\alpha}}\, d\mu(y)
$$

be the potential operator.

The following lemma was shown in [43] which follows from well-known arguments; we give the proof with slight modification for completeness of presentation and for convenience of the reader.

Lemma 4.4. *Let I^α be a potential operator, $1 < p < \infty$, $0 < \alpha < (1-\lambda)/p$, $0 \leqslant \lambda < 1$ and $1/p - 1/q = \alpha/(1-\lambda)$. If $b \in \mathrm{BMO}(X,\mu)$, then there exists a constant $C_{p,\alpha,\lambda} > 0$ such that for all functions f with compact support,*

$$
\|M([b,I^\alpha]f)\|_{L^{q,\lambda}(X,\mu)} \leqslant C_{p,q,\alpha,\lambda}\, \|b\|_{\mathrm{BMO}(X,\mu)}\, \|f\|_{L^{p,\lambda}(X,\mu)}, \tag{23}
$$

where

$$
C_{p,q,\alpha,\lambda} = C \left(b^{\lambda s/p}((p/s)')^{\frac{s}{p}}+1\right)^{1+\frac{p}{q}}\left(1+\frac{p}{1-\lambda-\alpha p}\right)[(p')^{1/q}+1]. \tag{24}
$$

Proof. For any ball $B = B(x,r) \subset X$ and any real number c, we write

$$
\begin{aligned}
[b,I_\alpha]f(y) &= [b-c,I_\alpha]f(y) \\
&= (b-c)I^\alpha f(y) - I^\alpha((b-c)f\chi_{c_0 B})(y) - I^\alpha\left((b-c)f\chi_{(c_0 B)^\complement}\right)(y) \\
&= \mathscr{I}_1(y) - \mathscr{I}_2(y) - \mathscr{I}_3(y),
\end{aligned}
$$

where c_0 is the constant depending on C_t and C_s, and will be determined later. Then, by the sublinearity of the maximal operator, we have

$$
M([b,I_\alpha]f)(x) \leqslant M\mathscr{I}_1(x) + M\mathscr{I}_2(x) + M\mathscr{I}_3(x).
$$

10

For $M\mathscr{I}_1(x)$, we have the pointwise estimate $M\mathscr{I}_1(x) \leqslant C \|b\|_{\mathrm{BMO}(X,\mu)} M_s(I^\alpha f(x))$ which follows from Hölder's inequality. Taking the boundedness of M_s and I^α into account, we have

$$
\begin{aligned}
\|M\mathscr{I}_1\|_{L^{q,\lambda}(X,\mu)} &\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|M_s(I^\alpha f)\|_{L^{q,\lambda}(X,\mu)} \\
&\lesssim (b^{\lambda s/p}((p/s)')^{\frac{s}{p}} + 1) \|b\|_{\mathrm{BMO}(X,\mu)} \|I^\alpha f\|_{L^{q,\lambda}(X,\mu)} \qquad (25) \\
&\lesssim C_{p,\alpha,\lambda} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)},
\end{aligned}
$$

where $C_{p,\alpha,\lambda}$ is (24).

For $1 < s < p < \infty$, $0 < \alpha < (1-\lambda)/p$, there exists $s_0, s_1, t_0, t > 1$, such that $1/s_0 = 1/t_0 - \alpha$, $1/t_0 = 1/s_1 + 1/s$ and $s/t = (\alpha p)/(1-\lambda)$. By Hölder's inequality together with Jensen's inequality and the fact that I^α is of strong type (t_0, s_0) we have (remembering that $B := B(x,r)$)

$$
\begin{aligned}
& M\mathscr{I}_2(x) \\
&\leq \sup_{0<r<d_X} \left(\fint_B |I_\alpha((b-c)f\chi_{c_0 B})(y)|^{s_0}\, d\mu(y) \right)^{1/s_0} \\
&\lesssim \sup_{0<r<d_X} \left(\frac{1}{\mu(B)^{1-t_0\alpha}} \int_{c_0 B} |b(y)-c|^{t_0}|f(y)|^{t_0}\, d\mu(y) \right)^{1/t_0} \\
&\lesssim \sup_{0<r<d_X} \left(\fint_{c_0 B} |b(y)-c|^{s_1}\, d\mu(y) \right)^{1/s_1} \left(\frac{1}{\mu(c_0 B)^{1-s\alpha}} \int_{c_0 B} |f(y)|^s\, d\mu(y) \right)^{1/s} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \sup_{0<r<d_X} \mu(c_0 B)^{\alpha-\frac{1}{s}} \left(\int_{c_0 B} |f(y)|^s\, d\mu(y) \right)^{\frac{1}{s}-\frac{1}{t}} \left[\mu(c_0 B)^{1-\frac{s}{p}} \left(\int_{c_0 B} |f(y)|^p d\mu(y) \right)^{s/p} \right]^{\frac{1}{t}} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \sup_{0<r<d_X} \mu(c_0 B)^{\alpha-\frac{1}{s}+\frac{1}{t}-\frac{s(1-\lambda)}{pt}} \left(\int_{c_0 B} |f(y)|^s\, d\mu(y) \right)^{\frac{1}{s}-\frac{1}{t}} \left(\frac{1}{\mu(c_0 B)^\lambda} \int_{c_0 B} |f(y)|^p\, d\mu(y) \right)^{\frac{s}{pt}} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} \sup_{0<r<d_X} \left(\fint_{c_0 B} |f(y)|^s\, d\mu(y) \right)^{\frac{1}{s}(1-\frac{s}{t})} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} (M_s f(x))^{1-\frac{\alpha p}{1-\lambda}}. \\
& \hspace{12cm} (26)
\end{aligned}
$$

Consequently, by Lemma 2.3,

$$
\begin{aligned}
\|M\mathscr{I}_2\|_{L^{q,\lambda}(X,\mu)} &\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} \left\| (M_s f)^{1-\frac{\alpha p}{1-\lambda}} \right\|_{L^{q,\lambda}(X,\mu)} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}^{\frac{\alpha p}{1-\lambda}} \|M_s f\|_{L^{p,\lambda}}^{p/q} \qquad (27) \\
&\lesssim (C_{p,\alpha,\lambda})^{p/q} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)},
\end{aligned}
$$

where $C_{p,\alpha,\lambda}$ is (24).

Since we have the validity of the reverse doubling condition, see (6), there exists constants $0 < \alpha, \beta < 1$ such that for all $x \in X$ and small positive r, $\mu B(x, \alpha r) \leqslant \beta \mu B(x, r)$. Let us take an integer m so that $\alpha^m d_X$ is sufficiently small.

Observe now that (see also [27, p. 929]) if $z \in B(x, r)$, then $B(x, r) \subset B(z, C_t(C_s + 1)r) \subset$

$B(x, C_t(C_t(C_s + 1) + 1)r)$ (rewrite it simply as $B(x,r) \subset B(z, c_1 r) \subset B(x, c_2 r)$). Hence,

$$
\begin{aligned}
\|M\mathscr{I}_3\|_{L^{q,\lambda}(X,\mu)} &\leqslant \sup_{\substack{x \in X \\ 0 < r < d_X}} \left(\frac{1}{\mu B(x,r)^\lambda} \int_{B(x,r)} |M(\mathscr{I}_3)(y)|^q \, d\mu(y) \right)^{\frac{1}{q}} \\
&\lesssim \sup_{\substack{x \in X \\ 0 < r < d_X}} \mu B(x,r)^{\frac{1-\lambda}{q}} \sup_{B \subset B(z,c_1 r)} \frac{1}{\mu B(z, c_1 r)} \int_{B(z,c_1 r)} |\mathscr{I}_3(y)| \, d\mu(y) \\
&\lesssim \sup_{B \subset B(z,c_1 r)} \mu B(z, \, c_1 r)^{\frac{1-\lambda}{q}-1} \int_{B(z,c_1 r)} |\mathscr{I}_3(y)| \, d\mu(y).
\end{aligned}
$$

Further, notice that when c_0 is an appropriate constant, $B \subset B(z, c_1 r)$, $y \in B(z, c_1 r)$, $\alpha^m d(y,z) \leq d(y,t) \leq \alpha^{m+1} d(y,z)$ and $z \in (c_0 B)^c$, then $d(x,t) > \bar{c}_0 r$, where $\bar{c}_0$ depends on C_t, C_s and c_0; it is also easy to see check that there are positive constants b_1, b_2 and b_3 such that $B(y, b_1 d(y,t)) \subset B(x, b_2 d(x,t)) \subset B(y, b_3 d(y,t))$. Consequently, by using Fubini's theorem, estimates of integrals (see Lemma 1.2 in [25]) we have for $y \in B(z, c_1 r)$,

$$
\begin{aligned}
\mathscr{I}_3(y) &\leqslant \int_{X \setminus c_0 B} |(b(z) - c)f(z)| \mu B(y, d(y,z))^{\alpha-1} \, d\mu(z) \\
&\leqslant C \int_{X \setminus c_0 B} |(b(z) - c)f(z)| \left(\int_{B(y, \alpha^m d(y,z)) \setminus B(y, \alpha^{m-1} d(y,z))} \mu B(y, d(y,t))^{\alpha-2} \, d\mu(t) \right) d\mu(z) \\
&\leqslant C \int_{X \setminus B(x, \bar{c}_0 r)} \mu B(y, d(y,t))^{\alpha-2} \left(\int_{B(y, a^{-m} d(y,t))} |(b(z) - c)f(z)| \, d\mu(z) \right) d\mu(t) \\
&\leqslant C \|b\|_{\mathrm{BMO}(X,\mu)} \int_{X \setminus B(x, \bar{c}_0 r)} \mu B(y, d(y,t))^{\alpha-1} \times \\
&\qquad\qquad\qquad\qquad \left(\frac{1}{\mu B(y, a^{1-m} d(y,t))} \int_{B(y, a^{1-m} d(y,t))} |f(z)|^p \, d\mu(z) \right)^{\frac{1}{p}} d\mu(t) \\
&\leqslant C \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)} \int_{X \setminus B(x, \bar{c}_0 r)} \mu B(y, d(y,t))^{\alpha - \frac{1-\lambda}{p}-1} \, d\mu(t) \\
&\leqslant C \mu B(x,r)^{\alpha - \frac{1-\lambda}{p}} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}.
\end{aligned}
$$

Thus applying the relation between $B(x,r)$ and $B(z,r)$ we find that

$$
\begin{aligned}
\|M\mathscr{I}_3\|_{L^{q,\lambda}(X,\mu)} &\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)} \sup_{B \subset B(z,c_1 r)} \mu B(z, \, c_1 r)^{\frac{1-\lambda}{q} + \alpha - \frac{1-\lambda}{p}} \\
&\lesssim \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}.
\end{aligned}
\tag{28}
$$

Gathering (25), (27), (28) it is easy to show that

$$
\|M([b, I^\alpha]f)\|_{L^{q,\lambda}(X,\mu)} \leqslant C_{p,q,\alpha,\lambda} \|b\|_{\mathrm{BMO}(X,\mu)} \|f\|_{L^{p,\lambda}(X,\mu)}. \qquad \square
$$

Before proving the next result, we define the following auxiliary functions which where introduced in [28].

Definition 4.5 (auxiliary functions). On an interval $(0, \delta]$, δ is small, we define the following functions:

$$\bar{\phi}(x) := p + \frac{(x-q)(1-\lambda+A_2(x))}{1-\lambda+A_2(x)-\alpha(x-q)}, \quad \tilde{\phi}(x) := q - \frac{(p-x)(1-\lambda+A_1(x))}{1-\lambda+A_1(x)-\alpha(p-x)}$$

$$\bar{A}(x) = 1 - \frac{\alpha(x-q)}{1-\lambda+A_2(x)}, \quad \tilde{A}(x) = \frac{1-\lambda+A_1(\eta)}{1-\lambda+A_1(\eta)-(p-\eta)\alpha}$$

$$\phi(x) := \bar{\phi}(x)^{\bar{A}(x)}, \quad \Phi(x) := \tilde{\phi}(x)^{\tilde{A}(x)}$$

$$\psi(\varepsilon) = \phi(\varepsilon^{\theta_1}), \quad \Psi(\varepsilon) = \Phi(\varepsilon^{\theta_1}),$$

for $\theta_1 > 0$.

Theorem 4.6. *Let I^α be a potential operator and let M be the maximal operator. Assume that $1 < p < \infty$, $0 < \alpha < (1-\lambda)/p$, $0 < \lambda < 1$, $1/p - 1/q = \alpha/(1-\lambda)$. Suppose that $\theta_1 > 0$ and that $\theta_2 \geqslant \theta_1[1 + \alpha q/(1-\lambda)]$. Let A_1 and A_2 be continuous non-negative functions on $(0, p-1]$ and $(0, q-1]$ respectively satisfying the conditions:*

  (i) $A_2 \in C^1((0,\delta])$ for some positive $\delta > 0$;

  (ii) $\lim_{x \to 0+} A_2(x) = 0$;

  (iii) $0 \leqslant B := \lim_{x \to 0+} \frac{\mathrm{d}}{\mathrm{d}x} A_2(x) < \frac{(1-\lambda)^2}{\alpha q^2}$;

  (iv) $A_1(\eta) = A_2(\bar{\phi}^{-1}(\eta))$, where $\bar{\phi}^{-1}$ is the inverse of $\bar{\phi}$ on $(0, \delta]$ for some $\delta > 0$.

  If $b \in \mathrm{BMO}(X, \mu)$, then the operator $M([b, I^\alpha])$ is bounded from $L^{p),\lambda}_{\theta_1, A_1}(X, \mu)$ to $L^{q),\lambda}_{\theta_2, A_2}(X, \mu)$.

Proof. We note that it is enough to prove the theorem for $\theta_2 = \theta_1(1 + \frac{\alpha q}{1-\lambda})$ because $\varepsilon^{\theta_2} \leqslant \varepsilon^{\theta_1(1+\frac{\alpha q}{1-\lambda})}$ for $\theta_2 > \theta_1[1 + (\alpha q)/(1-\lambda)]$ and small ε. We also note that, by L'Hospital rule, $\bar{\phi}(x) \sim x$ as $x \to 0+$ since $B < (1-\lambda)^2/(\alpha q^2)$. Moreover, $\bar{\phi}$ is invertible near 0, since $\frac{d\bar{\phi}}{dx}(x) > 0$. Under the conditions of Theorem 4.6 the function A_1 is continuous on $(0, \delta]$ and $\lim_{x \to 0+} A_1(x) = 0$. With all of the previous remarks taken into account, it is enough to prove the boundedness of $M([b, I^\alpha])$ from $L^{p),\lambda}_{\theta_1, A_1}(X, \mu)$ to $L^{q),\lambda}_{\psi, A_2}(X, \mu)$ since $\phi(x) \sim x^{1+\frac{\alpha q}{1-\lambda}}$, and consequently, $\psi(x) = \phi(x^{\theta_1}) \sim x^{\theta_1(1+\frac{\alpha q}{1-\lambda})}$ as $x \to 0$.

The case $\sigma < \varepsilon \leqslant s_{\max}$, where $s_{\max}$ is from (15). Letting

$$I := \psi^{\frac{1}{q-\varepsilon}}(\varepsilon)\left(\frac{1}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int_{B(x,r)}|M([b, I^\alpha]f)(y)|^{q-\varepsilon}\,\mathrm{d}\mu(y)\right)^{\frac{1}{q-\varepsilon}}$$

we have

$$I \lesssim \psi^{\frac{1}{q-\varepsilon}}(\varepsilon)\mu B(x,r)^{\frac{A_2(\varepsilon)+1-\lambda}{q-\varepsilon}}\left(\fint_{B(x,r)}|M([b, I^\alpha]f)(y)|^{q-\sigma}\,\mathrm{d}\mu(y)\right)^{\frac{1}{q-\sigma}}$$

$$\lesssim \psi^{\frac{1}{q-\varepsilon}}(\varepsilon)\mu B(x,r)^{\frac{A_2(\sigma)+1-\lambda}{q-\sigma}}\left(\fint_{B(x,r)}|M([b, I^\alpha]f)(y)|^{q-\sigma}\,\mathrm{d}\mu(y)\right)^{\frac{1}{q-\sigma}}$$

$$\lesssim \left(\sup_{\sigma \leqslant \varepsilon \leqslant s_{\max}}\psi^{\frac{1}{q-\varepsilon}}(\varepsilon)\right)\psi^{\frac{1}{\sigma-q}}(\sigma)\sup_{0<\varepsilon\leqslant\sigma}\sup_{\substack{x\in X\\r>0}}\left(\frac{\psi(\varepsilon)}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int_{B(x,r)}|M([b, I^\alpha]f)(y)|^{q-\varepsilon}\,\mathrm{d}\mu(y)\right)^{\frac{1}{q-\varepsilon}},$$

13

where the first inequality comes from Hölder's inequality and the second one is due to the fact that A_2 is bounded on $[\sigma, q-1)$ and $x \mapsto (1-\lambda)/(q-x)$ is an increasing function. Hence, it is enough to consider the case $0 < \varepsilon \leqslant \sigma$.

The case $0 < \varepsilon \leqslant \sigma$. Let η and ε be chosen so that

$$\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \frac{\alpha}{1-\lambda+A_2(\varepsilon)}. \tag{29}$$

Obviously we have that $\varepsilon \to 0$ if and only if $\eta \to 0$ and solving η with respect to ε in (29) we obtain

$$\eta = p - \frac{(q-\varepsilon)(1-\lambda+A_2(\varepsilon))}{1-\lambda+A_2(\varepsilon)-\alpha(\varepsilon-q)} = \bar{\phi}(\varepsilon).$$

Letting

$$J := \psi^{\frac{1}{q-\varepsilon}}(\varepsilon)\left(\frac{1}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int_{B(x,r)} |M([b, I^\alpha]f)(y)|^{q-\varepsilon}\, d\mu(y)\right)^{\frac{1}{q-\varepsilon}}$$

we have

$$J \lesssim C_{p-\eta,q-\varepsilon,\alpha,\lambda-A_2(\varepsilon)}\, \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \sup_{\substack{x\in X\\ r>0}}\left(\frac{1}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int_{B(x,r)} |f(y)|^{p-\eta}\, d\mu(y)\right)^{\frac{1}{p-\eta}}$$

$$\lesssim C_{p-\eta,q-\varepsilon,\alpha,\lambda-A_2(\varepsilon)}\, \eta^{\frac{\theta_1}{\eta-p}} \psi^{\frac{1}{q-\varepsilon}}(\varepsilon) \sup_{\substack{x\in X\\ r>0}}\left(\frac{\eta^{\theta_1}}{\mu B(x,r)^{\lambda-A_2(\varepsilon)}}\int_{B(x,r)} |f(y)|^{p-\eta}\, d\mu(y)\right)^{\frac{1}{p-\eta}}$$

$$\lesssim \|f\|_{L^{p),\lambda}_{\theta_1,A_1}(X,\mu)},$$

where the first inequality is due to Lemma 4.4 and the constant $C_{p-\eta,q-\varepsilon,\alpha,\lambda-A_2(\varepsilon)}$ is the one from (24). The last inequality is due to the fact that $\eta = \bar{\phi}(\varepsilon)$. Since the constant in the last inequality is uniformly bounded with respect to ε we obtain the desired boundedness of the operator. □

Corollary 4.7. *Let the conditions of Theorem 4.6 be fullfiled. Then the commutator $[b, I_\alpha]$ is bounded from $L^{p),\lambda}_{\theta_1,A_1}(X,\mu)$ to $L^{q),\lambda}_{\theta_2,A_2}(X,\mu)$.*

Proof. The result follows by the previous theorem and by the inequality

$$\|[b, I^\alpha]f\|_{L^{q),\lambda}_{\theta_2,A_2}(X,\mu)} \leqslant \|M([b, I^\alpha]f)\|_{L^{q),\lambda}_{\theta_2,A_2}(X,\mu)}. \qquad\qquad \square$$

5. Interior estimate of elliptic equations

In this section we apply the main result of this paper to establish some interior estimates of solutions to nondivergence elliptic equations with VMO coefficients. (see also the paper [32] for related topics). Suppose $n \geq 3$ and Ω is an open set in $\mathbb{R}^n$. Let

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)(\partial^2/\partial x_i \partial x_j),$$

where $a_{ij} = a_{ji}$ for $i, j = 1, 2, \cdots, n$, a.e. in Ω; assume that there exists $C > 0$ such that, for $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$,

$$C^{-1}|y|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)y_iy_j \leqslant C|y|^2, \text{ for a.e. } x \in \Omega;$$

denote by $(A_{ij})_{n\times n}$ the inverse of the matrix $(a_{ij})_{n\times n}$. For $x \in \Omega$ and $y \in \mathbb{R}^n$, let

$$K(x,\ y) = \frac{1}{\left[(n-2)C_n \sqrt{\det(a_{ij}(x))}\right]} \left(\sum_{i,j=1}^{n} A_{ij}(x)y_iy_j\right)^{1-n/2}$$

and $K_i(x,\ y) = \dfrac{\partial}{\partial y_i}K(x,\ y),\ K_{ij}(x,\ y) = \dfrac{\partial^2}{\partial x_i \partial x_j}K(x,\ y)$.

We denote by $VMO(\Omega)$ the class of all locally integrable functions with vanishing mean oscillation introduced in [40]. From [4, 9], we obtain the *interior representation formula*, that is, if $a_{ij} \in VMO \cap L^\infty(\Omega)$ and $u \in W_0^{2,r}(\Omega)$, $1 < r < \infty$ (see [4], [5],[20]),

$$u_{x_ix_j}(x) = \text{P.V.} \int_B K_{ij}(x, x-y)\left[\sum_{k,l=1}^{n}(a_{kl}(x) - a_{kl}(y))u_{x_kx_l}(y) + Lu(y)\right]dy$$

$$+ Lu(x)\int_{|y|=1} K_i(x,y)y_j\,d\delta_y,$$

a.e. for $x \in B \subset \Omega$, where B is a ball in Ω. We also set

$$M := \max_{i,j=1,\dots,n} \max_{|\alpha|\leqslant 2n} \left\|\partial^\alpha K_{ij}(x,y)/\partial y^\alpha\right\|_{L^\infty}.$$

To prove the next statement we need local version of Theorem 4.3 (see also Theorem 2.4 in [4] or Theorem 2.13 in [5]).

Corollary A. *Let* $1 < p < \infty$ *and let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Suppose that* $a \in VMO \cap L^\infty$. *Assume that* T *is the Calderón–Zygmund operator defined on* Ω *and that* η *is the VMO modulus of* a. *Then for any* $\varepsilon > 0$, *there exists a positive number* $\rho = \rho(\varepsilon, \eta)$ *such that for any balls* B_r *with the conditions:* $\Omega_r := B_r \cap \Omega \neq \emptyset$, $r \in (0, \rho)$ *and all* $f \in L_{\theta,A}^{p),\lambda}(\Omega_r)$ *the inequality*

$$\|[a,T]f\|_{L_{\theta,A}^{p),\lambda}(\Omega_r)} \leq C\varepsilon\|f\|_{L_{\theta,A}^{p),\lambda}(\Omega_r)}$$

is fulfilled.

Theorem 5.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *Suppose that* $1 < p,r < \infty$. *Let* $a_{ij} \in VMO(\Omega) \cap L^\infty$, $i,\ j = 1, 2, \cdots, n$. *Suppose that* $\eta_{i,j}$ *is the VMO modulus of* a_{ij}; *we set* $\eta = \left(\sum_{i,j=1}^{n} \eta_{i,j}\right)^{1/2}$. *Suppose also that* $M < \infty$. *Then there is a positive constant* $\rho = \rho(n, r, p, \lambda, M, \theta, A, \eta)$ *such that for all balls* $B \subset \Omega$ *with radius smaller than* ρ, *and* u *satisfying the conditions* $u \in W_0^{2,r}(\Omega)$, $\|Lu\|_{L_{\theta,A}^{p),\lambda}(B)} < \infty$ *we have that* $u_{x_ix_j} \in L_{\theta,A}^{p),\lambda}(B)$ *and, moreover, there exists a positive constant* $C = C(n, p, \lambda, \theta, M, A, \eta)$ *such that*

$$\left\|u_{x_ix_j}\right\|_{L_{\theta,A}^{p),\lambda}(B)} \leqslant C\,\|Lu\|_{L_{\theta,A}^{p),\lambda}(B)}.$$

Proof. It is easy to verify that K_{ij} satisfies the condition in Corollary by the representation of $u_{x_ix_j}$ and the conditions of K_{ij}. Thus, from Corollary A, we deduce, for any $\varepsilon > 0$,

$$\left\| u_{x_i x_j} \right\|_{L^{p),\lambda}_{\theta,A}(B)} \leqslant C\varepsilon \left\| u_{x_i x_j} \right\|_{L^{p),\lambda}_{\theta,A}(B)} + C \left\| Lu \right\|_{L^{p),\lambda}_{\theta,A}(B)}.$$

Choosing ε to be small enough (e.g. $\varepsilon < 1$), we then obtain

$$\left\| u_{x_i x_j} \right\|_{L^{p),\lambda}_{\theta,A}(B)} \leq (C/(1 - C\varepsilon)) \left\| Lu \right\|_{L^{p),\lambda}_{\theta,A}(B)}.$$

This finishes the proof. □

Acknowledgment

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